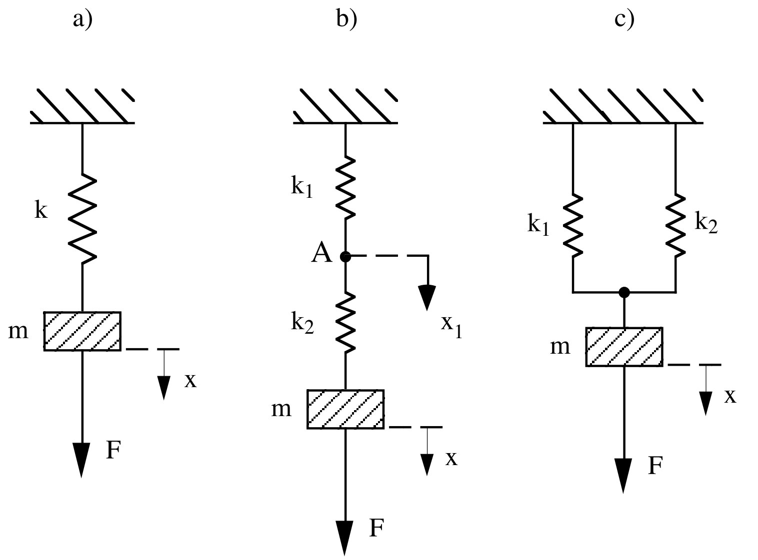
ME-221
SOLUTIONS FOR PROBLEM SET 2

Problem 1

At rest, $F = 0$ and as a result $x(0) = 0$.



a) $m\ddot{x} = F - kx$

b) $$m\ddot{x} = F - k_2(x - x_1) \quad (1)$$

Point A has zero mass:

$$0 = k_2(x - x_1) - k_1x_1 \Rightarrow x_1 = \frac{k_2}{k_1 + k_2}x \quad (2)$$

Combining (1) and (2):

$$m\ddot{x} = F - k_2\Big(x - \frac{k_2}{k_1 + k_2}x\Big) = F - \frac{k_1k_2}{k_1 + k_2}x$$

The equivalent spring constant k_{tot} for springs in series is given by $k_{tot} = \frac{k_1k_2}{k_1 + k_2}$ or $\frac{1}{k_{tot}} = \frac{1}{k_1} + \frac{1}{k_2}$

c) $m\ddot{x} = F - (k_1 + k_2)x$

The equivalent spring constant k_{tot} for springs in parallel is given by $k_{tot} = k_1 + k_2$

For serial spring arrangement, the total displacement is the sum of the displacements of each spring. The weakest spring is the most stretched. For parallel arrangement, the displacement is the same for each spring. We need higher force to achieve the same displacement in the presence of multiple springs.

Problem 2

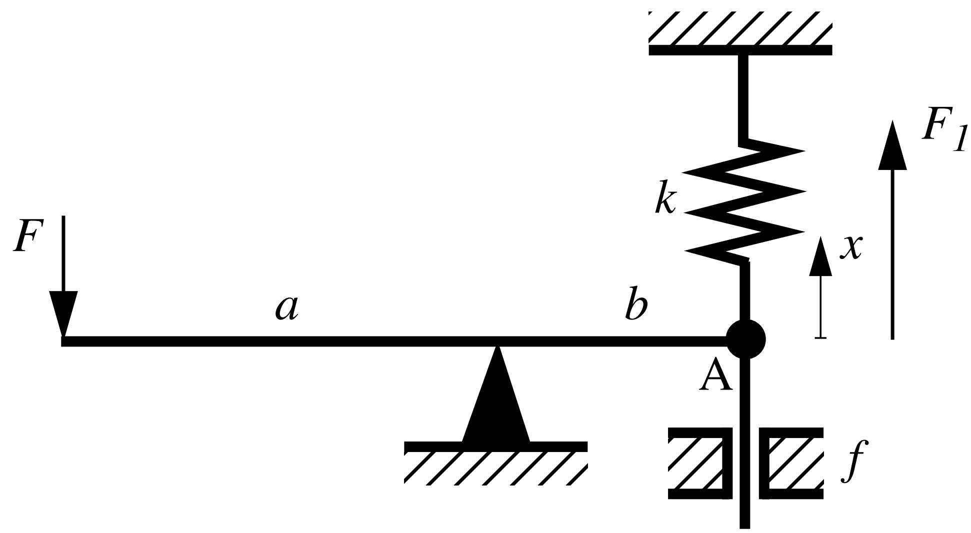


Figure 1: Spring-mass systems

First, we want to find the equivalent force of F at point A, which we can call F_1 as shown in the figure. For forces F and F_1 which stay aligned with x, the law of the lever gives:

$$Facos(\theta) = F_1bcos(\theta)$$

For small rotations we can assume that $cos(\theta) = 1$, therefore:

$$Fa = F_1b$$

At point A the equation of motion is:

$$m\ddot{x} = F_1 - kx - f\dot{x}$$

Assuming that the mass at point A is negligible, the final form of the equation would be:

$$kx + f\dot{x} - \frac{a}{b}F = 0$$

with $x(0) = 0$ for a system initially released while the lever arm is parallel to the ground.

The assumptions are 1) the rod and the spring are massless, 2) spring is linearly elastic, 3) lever is rigid and has no mass, and 4) the displacement x has small values.

Problem 3

a) Equations of motion are:

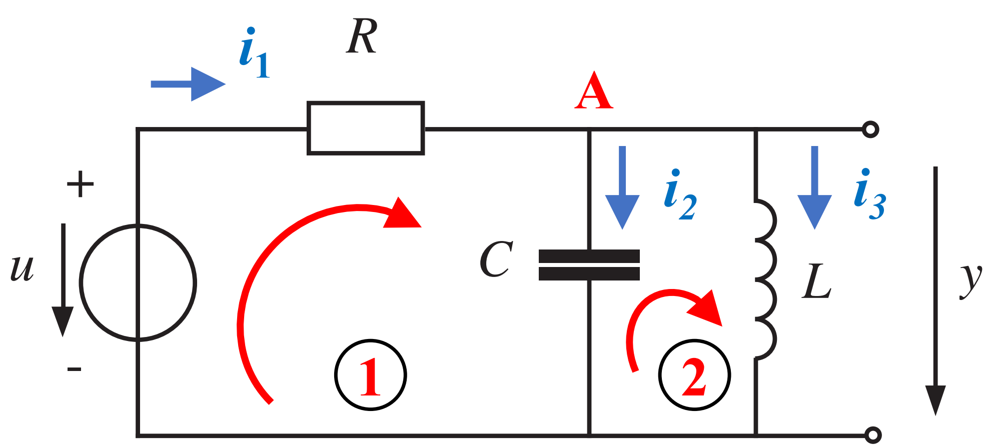
$$m\ddot{x} = kz + f\dot{z} \Rightarrow m(\ddot{y} - \ddot{z}) = kz + f\dot{z} \Rightarrow \ddot{z} + \frac{f}{m}\dot{z} + \frac{k}{m}z = \ddot{y}$$

with appropriate initial conditions such as $y(0) = \dot{y}(0) = 0$ and $z(0) = \dot{z}(0) = 0$

b) There are two special cases:

1. If k is large compared to m, f, and 1 then $\ddot{y} \sim z$.
2. If m is large compared to k, f, and 1 then $\ddot{y} \sim \ddot{z}$.

Problem 4



As a rule of thumb, the voltage on a capacitor and current passing through an inductor shall be selected as states of the system. Following this insight, states of the given circuit can be selected as $x_1 = y$ and $x_2 = i_3$

Kirchhoff's rule will give us three equations:

loop 1: $u - i_1R - x_1 = 0 \Rightarrow i_1 = \dfrac{u - x_1}{R}$

loop 2: $x_1 - y = 0$

node A: $i_1 = i_2 + x_2 \Rightarrow i_2 = \dfrac{u - x_1}{R} - x_2$

From the capacitance and inductance relations we obtain:

$$i_2 = \frac{d}{dt}(Cx_1) = \frac{d}{dt}((\alpha_0 + \alpha_1 x_1)x_1) = (\alpha_0 + 2\alpha_1 x_1)\dot{x}_1$$

$$x_1 = L\dot{x}_2$$

By replacing i_1 and i_2 with equivalent input and state formulation, we can write down state variables as a function of input and state variables:

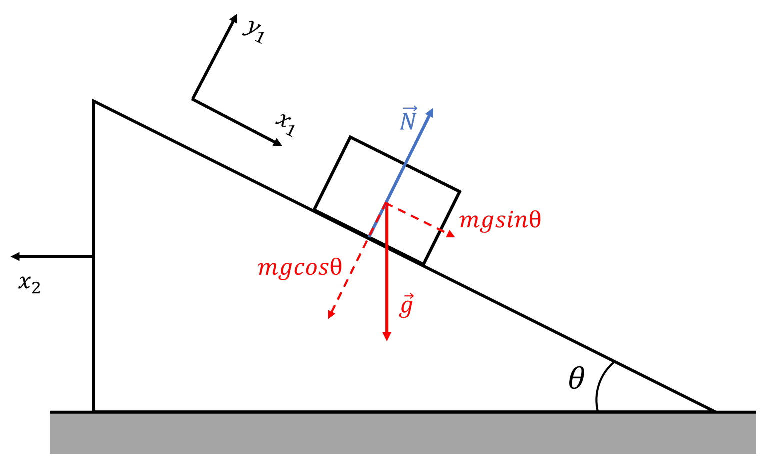
$$\dot{x}_1 = \frac{1}{\alpha_0 + 2\alpha_1 x_1}\left(\frac{u}{R} - \frac{x_1}{R} - x_2\right)$$

$$\dot{x_2} = \frac{1}{L}x_1$$

$$y = x_1$$

Problem 5

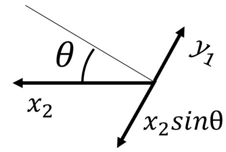
Let N be the normal force between the block and the inclined plane. The coordinate system of the block will be aligned with the inclined plane and that of the wedge with the expected direction of acceleration, as shown in the figure below:



The equations of motion are given by:

$$m\ddot{y}_1 = N - mgcos(\theta) \tag{1}$$
$$m\ddot{x}_1 = mgsin(\theta) \tag{2}$$
$$M\ddot{x}_2 = Nsin(\theta) \tag{3}$$

There are four unknowns ($\ddot{y}_1$, $\ddot{x}_1$, $\ddot{x}_2$, N) so we need one more equation. To satisfy the requirement that the block remains in contact with the inclined plane, we can impose that the displacement of the block in the direction perpendicular to that of the inclined plane must be equals and opposite to the displacement of the wedge in the same direction, as shown in the figure below.



Therefore we get:

$$y_1 = -x_2 sin(\theta) \tag{4}$$

By taking the second derivative, we get the fourth equation:

$$\ddot{y_1} = -\ddot{x}_2 sin(\theta) \tag{5}$$

Equations 1 and 3 can be rewritten as:

$$N = m\ddot{y_1} + mgcos(\theta) \tag{6}$$

$$N = \frac{M\ddot{x}_2}{sin(\theta)} \tag{7}$$

Solving for $\ddot{x}_2$ and by using the relation in equation 4, we get:

$$\ddot{x}_2 = \frac{mgsin(\theta)cos(\theta)}{M + msin^2(\theta)} \tag{8}$$